Demostrar que para $X \sim \mathcal{N}(\mu, \sigma^2)$, la entropía de X es

$$S(X) = \frac{1}{2}\log_2(2\pi e\sigma^2).$$

Tomando

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{1}$$

y

$$S(X) = -\int_{\mathbb{R}} f(x)\log_2 f(x)\,dx \tag{2}$$

para el caso continuo y sabiendo que

$$\log_b x = \frac{\log_a x}{\log_a b} \tag{3}$$

Entonces:

$$\begin{aligned}
S(X) = (1)(2) &= -\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \log_2\left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right) dx \\
&= -\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot \frac{1}{\log 2}\log\left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right) dx \\
&= -\frac{1}{\log 2}\left(\int_{\mathbb{R}} f(x)\log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) dx + \int_{\mathbb{R}} f(x)\cdot\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx\right) \\
&= -\frac{1}{\log 2}\left(\log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)\underbrace{\int_{\mathbb{R}} f(x)\,dx}_{=1} - \frac{1}{2\sigma^2}\underbrace{\int_{\mathbb{R}} f(x)(x-\mu)^2\,dx}_{=\mathbb{E}[(X-\mu)^2]}\right) \\
&= \frac{-\log\left(\frac{1}{\sqrt{2\pi}\sigma}\right)}{\log 2} + \underbrace{\frac{\sigma^2}{2\sigma^2\log 2}}_{\frac{1}{2\log 2}} \\
&= \frac{-\log 1 + \log(\sqrt{2\pi}\sigma)}{\log 2} + \frac{1}{2\log 2} \\
&= \frac{\log(\sqrt{2\pi}\sigma)}{\log 2} + \frac{1}{2\log 2} \\
&= \log_2(\sqrt{2\pi}\sigma) + \frac{\log e}{2\log 2} \\
&= \log_2(\sqrt{2\pi}\sigma) + \frac{1}{2}\log_2 e \\
&= \frac{1}{2}\left(\log_2\left((\sqrt{2\pi}\sigma)^2\right) + \log_2 e\right) \\
&= \frac{1}{2}\log_2(2\pi\sigma^2 e)
\end{aligned}$$

De esta manera se puede observar que la entropía no depende de la media de la gaussiana sino de la varianza.